Europhotonics : Questions in Quantum Optics

1. Demonstrate the identity :

$$\left[\widehat{A}, \widehat{B}\widehat{C}\right] = \left[\widehat{A}, \widehat{B}\right]\widehat{C} + \widehat{B}\left[\widehat{A}, \widehat{C}\right] \tag{1}$$

Solution :

$$\left[\widehat{A}, \widehat{B}\widehat{C}\right] = \widehat{A}\widehat{B}\widehat{C} - \widehat{B}\widehat{C}\widehat{A}$$
$$\left[\widehat{A}, \widehat{B}\right]\widehat{C} + \widehat{B}\left[\widehat{A}, \widehat{C}\right] = \widehat{A}\widehat{B}\widehat{C} - \widehat{B}\widehat{A}\widehat{C} + \widehat{B}\widehat{A}\widehat{C} - \widehat{B}\widehat{C}\widehat{A}\,.$$

2. Use the identity of eq. (1) to evaluate the following expressions : (Rappel : $\widehat{N}_\ell = \widehat{a}_\ell^\dagger \widehat{a}_\ell$ and $\left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger\right] = 1$),

A. $\left[\widehat{a}_\ell, \widehat{N}_\ell\right]$

Solution :

$$\left[\widehat{a}_\ell, \widehat{N}_\ell\right] = \left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger \widehat{a}_\ell\right] = \left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger\right]\widehat{a}_\ell + \widehat{a}_\ell^\dagger\left[\widehat{a}_\ell, \widehat{a}_\ell\right]$$
$$= \widehat{a}_\ell$$

B. $\left[\widehat{a}_\ell^\dagger, \widehat{N}_\ell\right]$

Solution :

$$\left[\widehat{a}_\ell^\dagger, \widehat{N}_\ell\right] = \left[\widehat{a}_\ell^\dagger, \widehat{a}_\ell^\dagger \widehat{a}_\ell\right] = \left[\widehat{a}_\ell^\dagger, \widehat{a}_\ell^\dagger\right]\widehat{a}_\ell + \widehat{a}_\ell^\dagger\left[\widehat{a}_\ell^\dagger, \widehat{a}_\ell\right]$$
$$= -\widehat{a}_\ell^\dagger$$

C. $\left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger \widehat{a}_\ell \widehat{a}_\ell^\dagger\right]$

Solution :

$$\left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger \widehat{a}_\ell \widehat{a}_\ell^\dagger\right] = \left[\widehat{a}_\ell, \widehat{N}_\ell\right]\widehat{a}_\ell^\dagger + \widehat{N}_\ell\left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger\right] = \widehat{a}_\ell \widehat{a}_\ell^\dagger + \widehat{N}_\ell$$
$$= 1 + 2\widehat{a}_\ell^\dagger \widehat{a}_\ell = 1 + 2\widehat{N}_\ell\,.$$

For the following questions, let us consider the special (but common case) where $\left[\widehat{A}, \widehat{B}\right] \neq 0$ but we still have the condition that :

$$\left[\widehat{A}, \left[\widehat{A}, \widehat{B}\right]\right] = 0 = \left[\widehat{B}, \left[\widehat{A}, \widehat{B}\right]\right]\,. \tag{2}$$

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3. Under the condition of eq. (2), demonstrate the identity :

$$\left[\widehat{B}, \widehat{A}^n\right] = n\widehat{A}^{n-1}\left[\widehat{B}, \widehat{A}\right]\,. \tag{3}$$

Solution : We can prove this relation by recurrence. It is trivially true $n = 1$:

$$\left[\widehat{B}, \widehat{A}^{n=1}\right] = \widehat{A}^0\left[\widehat{B}, \widehat{A}\right] = \left[\widehat{B}, \widehat{A}\right]$$

Provided that eq. (3) is true for any $n \geq 1$, then we can show that it is true for $n + 1$:

$$\left[\widehat{B}, \widehat{A}^{n+1}\right] = \left[\widehat{B}, \widehat{A}^n \widehat{A}\right] = \left[\widehat{B}, \widehat{A}^n\right]\widehat{A} + \widehat{A}^n\left[\widehat{A}, \widehat{B}\right]$$
$$= n\widehat{A}^{n-1}\left[\widehat{B}, \widehat{A}\right]\widehat{A} + \widehat{A}^n\left[\widehat{B}, \widehat{A}\right] \tag{4}$$

where we used eq. (1) and eq. (3). With the precondition of eq. (2) we have that $\left[\widehat{A},\left[\widehat{A},\widehat{B}\right]\right]=0$, which tells us that :

$$\left[\widehat{A},\left[\widehat{B},\widehat{A}\right]\right]=0 \Longrightarrow \left[\widehat{B},\widehat{A}\right]\widehat{A}=\widehat{A}\left[\widehat{B},\widehat{A}\right] \tag{5}$$

Putting eq. (5) into eq. (4) we find :

$$\begin{aligned}\left[\widehat{B},\widehat{A}^{n+1}\right]=\left[\widehat{B},\widehat{A}^{n}\widehat{A}\right]&=\left[\widehat{B},\widehat{A}^{n}\right]\widehat{A}+\widehat{A}^{n}\left[\widehat{B},\widehat{A}\right]\\ &=n\widehat{A}^{n-1}\widehat{A}\left[\widehat{B},\widehat{A}\right]+\widehat{A}^{n}\left[\widehat{B},\widehat{A}\right]\\ &=(n+1)\,\widehat{A}^{n}\left[\widehat{B},\widehat{A}\right] \qquad \text{Q.E.D.}\end{aligned} \tag{6}$$

4. Use the identity derived in eq. (3) to show that :

$$\left[\widehat{B},e^{-\widehat{A}x}\right]=-xe^{-\widehat{A}x}\left[\widehat{B},\widehat{A}\right]\ . \tag{7}$$

Solution :

$$\begin{aligned}\left[\widehat{B},e^{-\widehat{A}x}\right]&=\sum_{m=0}\frac{1}{m!}\left[\widehat{B},\left(-\widehat{A}x\right)^{m}\right]=\sum_{n=1}\frac{x^{m}(-1)^{m}}{m!}\left[\widehat{B},\widehat{A}^{m}\right]\\ &=\sum_{n=1}\frac{x^{m}(-1)^{m}}{m!}m\widehat{A}^{m-1}\left[\widehat{B},\widehat{A}\right]\\ &=\sum_{m=1}(-x)\frac{(-1)^{m-1}x^{m-1}\widehat{A}^{m-1}}{(m-1)!}\left[\widehat{B},\widehat{A}\right]=\sum_{n=0}(-x)\frac{\left(-x\widehat{A}\right)^{n}}{n!}\left[\widehat{B},\widehat{A}\right]\\ &=-xe^{-\widehat{A}x}\left[\widehat{B},\widehat{A}\right]\ ,\end{aligned}$$

where we have invoked the identity of eq. (3) to arrive at the second and we make a change of index, $n\equiv m-1$ at the end of the third line.

5. Use the identity eq. (7) to derive the following expression :

$$e^{\widehat{A}x}\widehat{B}e^{-\widehat{A}x}=\widehat{B}-x\left[\widehat{B},\widehat{A}\right]\ . \tag{8}$$

Solution : We multiply eq. (7) on the left by $e^{\widehat{A}x}$

$$e^{\widehat{A}x}\left[\widehat{B},e^{-\widehat{A}x}\right]=e^{\widehat{A}x}\widehat{B}e^{-\widehat{A}x}-e^{\widehat{A}x}e^{-\widehat{A}x}\widehat{B}=-xe^{\widehat{A}x}e^{-\widehat{A}x}\left[\widehat{B},\widehat{A}\right] \tag{9}$$

$$e^{\widehat{A}x}\widehat{B}e^{-\widehat{A}x}=\widehat{B}-x\left[\widehat{B},\widehat{A}\right] \tag{10}$$

6. Let us define the operator, $\widehat{O}(x)\equiv e^{\widehat{A}x}e^{\widehat{B}x}$. Calculate the derivative : $\frac{d\widehat{O}}{dx}$, and use the expression of eq. (8) to show that :

$$\frac{d\widehat{O}}{dx}=\left(\widehat{A}+\widehat{B}+x\left[\widehat{A},\widehat{B}\right]\right)\widehat{O}\ . \tag{11}$$

Solution :

$$\begin{aligned}\frac{d\widehat{O}}{dx}&=\frac{d}{dx}e^{\widehat{A}x}e^{\widehat{B}x}=\widehat{A}e^{\widehat{A}x}e^{\widehat{B}x}+e^{\widehat{A}x}\widehat{B}e^{\widehat{B}x}\\ &=\widehat{A}e^{\widehat{A}x}e^{\widehat{B}x}+e^{\widehat{A}x}\widehat{B}e^{-\widehat{A}x}e^{\widehat{A}x}e^{\widehat{B}x}\\ &=\left\{\widehat{A}+e^{\widehat{A}x}\widehat{B}e^{-\widehat{A}x}\right\}\widehat{O}\\ &=\left(\widehat{A}+\widehat{B}+x\left[\widehat{A},\widehat{B}\right]\right)\widehat{O}\end{aligned} \tag{12}$$

where we used eq. (8) in the last line.

7. Since $\left[\widehat{A},\widehat{B}\right]$ commutes with both $\widehat{A}$ and $\widehat{B}$, we can solve eq. (11) with the usual solution to differential equations methodology. In this manner, derive the relation (known as the Baker-Haussdorf-Campbell theorem) :

$$e^{\widehat{A}}e^{\widehat{B}} = e^{\widehat{A}+\widehat{B}+\frac{1}{2}[\widehat{A},\widehat{B}]} \ . \tag{13}$$

Solution : Since eq. (11) can be treated as an ordinary differential equation involving 'classical' numbers, the solution is found by integrating a separable equation :

$$\int \frac{d\widehat{O}}{\widehat{O}} = \int \left(\widehat{A}+\widehat{B}+x\left[\widehat{A},\widehat{B}\right]\right)dx$$

$$\Longrightarrow \ln\widehat{O} = \left(\widehat{A}x+\widehat{B}x+\frac{x^2}{2}\left[\widehat{A},\widehat{B}\right]\right)+\text{Cte} \tag{14}$$

$$\Longrightarrow \widehat{O}(x) = C\exp\left(\widehat{A}x+\widehat{B}x+\frac{x^2}{2}\left[\widehat{A},\widehat{B}\right]\right) \tag{15}$$

where we must have $C=1$ in order satisfy the limit condition at $x=0$, and we finally obtain eq. (13) theorem by taking $x=1$. The result of eq.(13) is in fact a special case of a more general formulahttps://en.wikipedia.org/wiki/Baker-Campbell-Hausdorff_formula.

8. Let us consider the following harmonic oscillator state (or alternatively a mono-mode electromagnetic state) :

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left\{|0\rangle+|4\rangle\right\} \ . \tag{16}$$

A. Calculate $\overline{n} = \langle\Psi|\,\widehat{N}|\Psi\rangle = \langle\Psi|\,\widehat{a}^\dagger\widehat{a}|\Psi\rangle$.

Solution :

$$\langle\Psi|\,\overline{n}|\Psi\rangle = \frac{1}{2}\left\{\langle 4|+\langle 0|\right\}\widehat{a}^\dagger\widehat{a}\left\{|0\rangle+|4\rangle\right\} = \frac{1}{2}\left\{\langle 4|+\langle 0|\right\}\widehat{N}\left\{|0\rangle+|4\rangle\right\} = 2$$

B. Calculate $\overline{n^2} = \langle\Psi|\,\widehat{N}^2|\Psi\rangle$.

Solution :

$$\langle\Psi|\,\widehat{N}^2|\Psi\rangle = \frac{1}{2}\left\{\langle 4|+\langle 0|\right\}\widehat{N}^2\left\{|0\rangle+|4\rangle\right\} = 8$$

.

C. Calculate $\Delta n \equiv \sqrt{\overline{n^2}-\overline{n}^2}$

Solution :

$$\Delta n = \sqrt{\overline{n^2}-\overline{n}^2} = 2$$

Let us continue to consider mono-mode radiation states. The quadrature operators of the mode are :

$$X_1 = \frac{1}{\sqrt{2}}\left(\widehat{a}+\widehat{a}^\dagger\right)$$
$$X_2 = \frac{1}{i\sqrt{2}}\left(\widehat{a}-\widehat{a}^\dagger\right) \tag{17}$$

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9. For the mono-mode vacuum state, $|0\rangle$, and the quadrature operators , X_1, and X_2, given in eq. (17) :

A. Calculate $\langle X_i\rangle_0 = \langle 0|\,X_i|0\rangle$, for $i=1,2$.

Solution :

$$\langle X_i\rangle_0 = \frac{1}{\sqrt{2}}\langle 0|\left(\widehat{a}+\widehat{a}^\dagger\right)|0\rangle = \frac{1}{i\sqrt{2}}\langle 0|\left(\widehat{a}-\widehat{a}^\dagger\right)|0\rangle = 0$$

B. Calculate $\left\langle X_i^2 \right\rangle_0 = \langle 0 | \, X_i^2 | 0 \rangle$, for $i = 1, 2$.

Solution : Taking the square of the quadrature operators, and using the fact that :

$$\left[\widehat{a}, \widehat{a}^\dagger\right] = 1 \Longrightarrow \widehat{a}\widehat{a}^\dagger = 1 + \widehat{a}^\dagger\widehat{a}$$

we find that

$$\begin{aligned} \widehat{X}_1^2 &= \frac{1}{2}\left(\widehat{a}\widehat{a} + \widehat{a}^\dagger\widehat{a}^\dagger + 2\widehat{a}^\dagger\widehat{a} + 1\right) \\ \widehat{X}_2^2 &= -\frac{1}{2}\left(\widehat{a}\widehat{a} + \widehat{a}^\dagger\widehat{a}^\dagger - 2\widehat{a}^\dagger\widehat{a} - 1\right) \end{aligned} \tag{18}$$

$$\Longrightarrow \langle 0|\widehat{X}_1^2|0\rangle = \langle 0|\widehat{X}_2^2|0\rangle = \frac{1}{2} \, .$$

C. Calculate $\Delta X_i = \sqrt{\langle X_i^2 \rangle_0 - \langle X_i \rangle_0^2}$ for $i = 1, 2$.

Solution : We have

$$\Delta X_i = \frac{1}{\sqrt{2}} \qquad \text{for } i = 1, 2 \tag{19}$$

This means that there are always fluctuations of the dynamic variables in the vacuum state (a signature result of quantum mechanics). N.b. that this does not mean that everything is uncertain in quantum mechanics (quantities like energy, polarization, and angular momentum may be perfectly well-defined for example). One should remark that by adopting dimensionless quadrature operators, we are effectively working in 'natural' units of $\hbar = 1$, and the Heisenberg uncertainty relations read $\Delta X_1 \Delta X_2 \geq \frac{1}{2}$. The vacuum state thereby corresponds to a limit of the minimal Heisenberg uncertainty, $\Delta X_1 \Delta X_2 = \frac{1}{2}$. It is also worth remarking that there is equal uncertainty in the X_1 and X_2 coordinates, (*i.e.* $\Delta X_1 = \Delta X_2 = 1/\sqrt{2}$). This equality may seem rather 'obvious' given the symmetry of the vacuum state, but we should ask ourselves if the equality of uncertainty in X_1 and X_2 always holds (the next question will give us an example of situations where $\Delta X_1 \neq \Delta X_2$ and where one can have $\Delta X_1 < 1/\sqrt{2}$ provided that $\Delta X_2 > 1/\sqrt{2}$, or vice versa).

Let us now consider a mono-mode photon state, $|\Psi\rangle$, which is a superposition of the vacuum, $|0\rangle$ and a 1-photon state, $|1\rangle$. We can then adopt the 'qubit' notation for this state and write an arbitrary superposition as :

$$|\Psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle \, . \tag{20}$$

where $\theta \in [0, 2\pi]$ and $\phi \in [0, 2\pi[$. **N.b.** the states $|0\rangle$ and $|1\rangle$ are related by the usual 'ladder' operators $\widehat{a}^\dagger|0\rangle = |1\rangle$, $\widehat{a}|1\rangle = |0\rangle$.

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10. For the state $|\Psi\rangle$ of eq. (20) :

A. Calculate $\langle X_i \rangle_\Psi = \langle \Psi | \, X_i | \Psi \rangle$, for $i = 1, 2$

Solution :

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{2}}\left\{\langle 0|\cos(\theta/2) + \langle 1|e^{-i\phi}\sin(\theta/2)\right\}\left(\widehat{a} + \widehat{a}^\dagger\right)\left\{\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right\} \\ &= \frac{2\sin(\theta/2)\cos(\theta/2)}{\sqrt{2}}\frac{\left(e^{i\phi} + e^{-i\phi}\right)}{2} = \sqrt{2}\sin(\theta/2)\cos(\theta/2)\cos\phi \end{aligned}$$

$$\begin{aligned} X_2 &= \frac{1}{i\sqrt{2}}\left\{\langle 0|\cos(\theta/2) + \langle 1|e^{-i\phi}\sin(\theta/2)\right\}\left(\widehat{a} - \widehat{a}^\dagger\right)\left\{\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right\} \\ &= \frac{2\sin(\theta/2)\cos(\theta/2)}{i\sqrt{2}}\frac{\left(e^{i\phi} - e^{-i\phi}\right)}{2} = \sqrt{2}\sin(\theta/2)\cos(\theta/2)\sin\phi \end{aligned}$$

B. Calculate $\left\langle X_i^2 \right\rangle_\Psi \equiv \langle \Psi | \, X_i^2 | \Psi \rangle$, for $i = 1, 2$.

Solution :

$$\begin{aligned} X_1^2 &= \frac{1}{2}\left\{\langle 0|\cos(\theta/2) + \langle 1|e^{-i\phi}\sin(\theta/2)\right\}\left(\widehat{a}\widehat{a} + \widehat{a}^\dagger\widehat{a}^\dagger + 2\widehat{a}^\dagger\widehat{a} + 1\right)\left\{\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right\} \\ &= \sin^2(\theta/2) + \frac{1}{2} \end{aligned}$$

$$X_2^2 = \frac{1}{2}\left\{\langle 0|\cos(\theta/2) + \langle 1|e^{-i\phi}\sin(\theta/2)\right\}\left(-\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger + 2\hat{a}^\dagger\hat{a} + 1\right)\left\{\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right\}$$
$$= \sin^2(\theta/2) + \frac{1}{2}$$

C. Calculate $\Delta X_i(\theta,\phi) = \sqrt{\langle X_i^2\rangle_0 - \langle X_i\rangle_0^2}$ for $i = 1, 2$.

Solution :

$$\Delta X_1(\theta,\phi) = \left[\sin^2(\theta/2)\left(1 - 2\cos^2(\theta/2)\cos^2\varphi\right) + \frac{1}{2}\right]^{1/2} \tag{21a}$$
$$\Delta X_2(\theta,\phi) = \left[\sin^2(\theta/2)\left(1 - 2\cos^2(\theta/2)\sin^2\varphi\right) + \frac{1}{2}\right]^{1/2} . \tag{21b}$$

D. Calculate $\Delta X_1\Delta X_2$ as a function of θ (and an arbitrary value of φ since $\Delta X_1\Delta X_2$ is φ independent as you can verify), and plot $\Delta X_1\Delta X_2$ as a function of $\theta \in \{0, 360\}$. Are the Heisenberg uncertainties satisfied ? Explain the behavior of this plot.

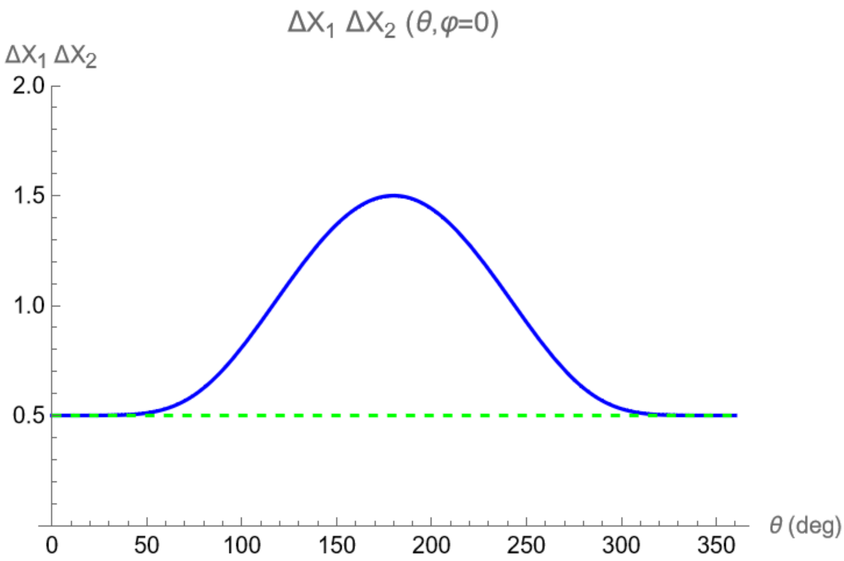

FIGURE 1 – $\Delta X_1\Delta X_2(\theta)$ for the state $|\Psi\rangle$ of eq. (20).

Solution : We see from fig. 1 that the minimal limit of the Heisenberg uncertainty is obtained when $|\Psi\rangle$ is the pure ground state, *i.e.* $\theta = 0$ or 2π. Uncertainty reaches a maximum here for $\theta = 180$ which corresponds to $|\Psi\rangle$ being a pure $|1\rangle$ state. This is expected since pure number states, excepting the vacuum have more than minimal quantum fluctuations, $\Delta X_1\Delta X_2 \geq 1/2$.

E. Calculate $\Delta X_1(\theta,\phi)$ and $\Delta X_2(\theta,\phi)$ as functions of θ for $\phi = 0$ and $\phi = \pi/2$ with $\theta \in \{0, 360°\}$. Can $\Delta X_i(\theta,\phi)$ have values inferior to those found for the vacuum state in the previous question ? Explain the significance of this.

Solution : We see from fig. 2 that for certain values of θ, $\Delta X_1(\theta)$ or $\Delta X_2(\theta)$ may individually go below uncertainties of, $\frac{1}{\sqrt{2}}$, found for the vacuum state. Nevertheless, it is important to remark this this is always accompanied by increased uncertainty in the conjugate variable such that the Heisenberg quantum uncertainties, plotted in fig. 1 remain satisfied. Nowadays, such states are frequently called 'squeezed states' since the fluctuations of a quadrature operator for such a state can be less than $1/\sqrt{2}$.
The state $|\Psi\rangle$ in the present case is limited to a superposition of only 2 states of light. Typical 'squeezed states' of light that one sees in the literature are constructed as an infinite superposition of of number states, (in a fashion quite analogous to the 'coherent states' considered in the following questions).

For the the following questions, let us recall that an arbitrary monomode photon state, $|\psi\rangle$, can be written as a superposition of number states (Fock states), $|n\rangle$:

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n\,|n\rangle \ . \tag{22}$$

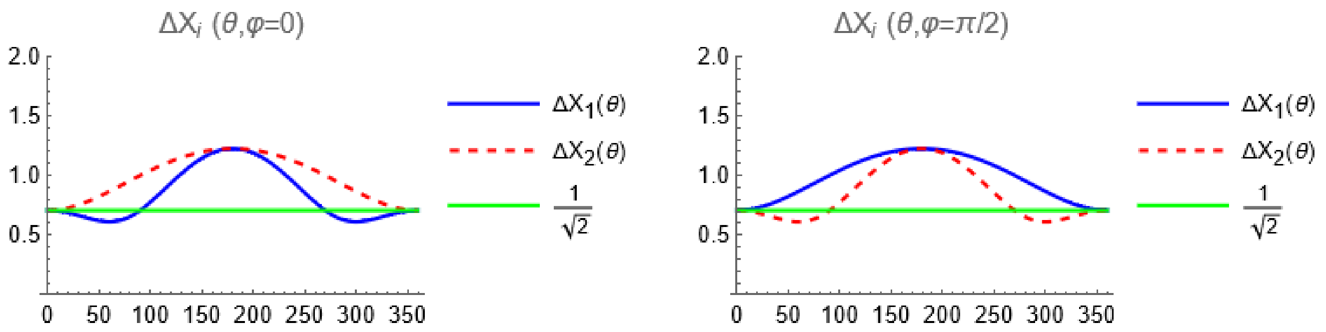

FIGURE 2 – Quantum uncertainties, $\Delta X_1(\theta)$ and $\Delta X_2(\theta)$ for the state $|\Psi\rangle$ of eq. (20).

A quasi-classical state (Glauber state/coherent state) is given by the following superposition of number states :

$$|\alpha\rangle \equiv e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{23}$$

This expansion has the property (and can be deduced from the fact) that $|\alpha\rangle$ is an eigenstate of the destruction operator :

$$\widehat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{24}$$

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11. Demonstrate that $|\alpha\rangle$ is an eigenstate of the operator $\widehat{a}$.

Solution :

$$\widehat{a}\,|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{\nu=0}^{\infty} \frac{\alpha^\nu}{\sqrt{\nu!}} \widehat{a}\,|\nu\rangle = e^{-|\alpha|^2/2} \sum_{\nu=1}^{\infty} \frac{\alpha^\nu}{\sqrt{\nu!}} \sqrt{\nu}\,|\nu-1\rangle = \alpha e^{-|\alpha|^2/2} \sum_{\nu=1}^{\infty} \frac{\alpha^{\nu-1}}{\sqrt{(\nu-1)!}} |\nu-1\rangle$$
$$= \alpha e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \alpha|\alpha\rangle \ ,$$

where in the last line we made the substitution of a new 'dummy' index, $n \equiv \nu - 1$

12. Demonstrate that one cannot construct an eigenstates of the operator $\widehat{a}^\dagger$. In other words, if $\widehat{a}^\dagger|\beta\rangle = \beta|\beta\rangle$ then $|\beta\rangle$ is the null vector (not to be confused with the vacuum state, $|0\rangle$), i.e. $|\beta\rangle = 0$.

Solution : If the state $|\beta\rangle$ exists then we know that it can be expressed as a superposition of number states :

$$|\beta\rangle = \sum_{n=0}^{\infty} C_n\,|n\rangle \ . \tag{25}$$

Acting on this state with a creation operator yields :

$$\widehat{a}^\dagger\,|\beta\rangle = \sum_{\nu=0}^{\infty} C_\nu \widehat{a}^\dagger\,|\nu\rangle = \sum_{\nu=0}^{\infty} C_\nu \sqrt{\nu+1}\,|\nu+1\rangle = \sum_{\nu=0}^{\infty} C_\nu \sqrt{\nu+1}\,|\nu+1\rangle = \sum_{n=1}^{\infty} C_{n-1}\sqrt{n}\,|n\rangle \ . \tag{26}$$

The eigenvalue expression then requires that :

$$\widehat{a}^\dagger\,|\beta\rangle = \sum_{n=1}^{\infty} C_{n-1}\sqrt{n}\,|n\rangle = \beta \sum_{n=0}^{\infty} C_n\,|n\rangle \ . \tag{27}$$

which requires that

$$\begin{aligned} C_0 &= 0 \\ C_n &= \frac{\sqrt{n}}{\beta} C_{n-1} \qquad n \in \{1, 2, ...\infty\} \\ \Longrightarrow C_n &= 0 \qquad n \in \{1, 2, ...\infty\} \\ \Longrightarrow |\beta\rangle &= 0 \ . \end{aligned} \tag{28}$$

13. For a coherent state, $|\alpha\rangle$.

A. Calculate the average number of photons : $\overline{n} = \langle \widehat{n} \rangle \equiv \langle \alpha | \, \widehat{a}^\dagger \widehat{a} \, |\alpha\rangle$.

Solution :

$$\begin{aligned}\overline{n} = \langle \widehat{n} \rangle \equiv \langle \alpha | \, \widehat{a}^\dagger \widehat{a} \, |\alpha\rangle &= \langle \alpha | \, \alpha^* \alpha \, |\alpha\rangle \\ &= \langle \alpha | \, |\alpha|^2 \, |\alpha\rangle \\ &= |\alpha|^2 \ ,\end{aligned}$$

where we used the property of eq. (24) and its adjoint, $\langle \alpha | a^\dagger = \langle \alpha | \alpha^*$.

B. Calculate $\langle X_i \rangle_\alpha = \langle \alpha | \, X_i | \alpha \rangle$ for $i = 1, 2$.

Solution :

$$\begin{aligned}\langle \alpha | X_1 | \alpha \rangle &= \frac{1}{\sqrt{2}} \langle \alpha | \left(\widehat{a} + \widehat{a}^\dagger \right) |\alpha\rangle \\ &= \frac{1}{\sqrt{2}} \left(\alpha + \alpha^* \right) = \sqrt{2} \Re \{\alpha\}\end{aligned} \tag{29a}$$

$$\begin{aligned}\langle \alpha | X_2 | \alpha \rangle &= \frac{1}{i\sqrt{2}} \langle \alpha | \left(\widehat{a} - \widehat{a}^\dagger \right) |\alpha\rangle \\ &= \frac{1}{i\sqrt{2}} \left(\alpha - \alpha^* \right) = \sqrt{2} \Im \{\alpha\}\end{aligned} \tag{29b}$$

C. Calculate $\left\langle X_i^2 \right\rangle_\alpha \equiv \langle \alpha | \, X_i^2 | \alpha \rangle$, for $i = 1, 2$.

Solution :

$$\begin{aligned}\left\langle \widehat{X}_1^2 \right\rangle &= \frac{1}{2} \langle \alpha | \, \widehat{a}\widehat{a} + \widehat{a}^\dagger \widehat{a}^\dagger + 2\widehat{a}^\dagger \widehat{a} + 1 \, |\alpha\rangle \\ &= \frac{1}{2} \left\{ \alpha^2 + (\alpha^*)^2 + 2\alpha^* \alpha + 1 \right\} \\ &= \frac{1}{2} \left(\alpha + \alpha^* \right)^2 + \frac{1}{2} = 2\Re \{\alpha\}^2 + \frac{1}{2} \ ,\end{aligned} \tag{30a}$$

where we have used :

$$\left[\widehat{a}, \widehat{a}^\dagger \right] = 1 \Longrightarrow \widehat{a}\widehat{a}^\dagger = 1 + \widehat{a}^\dagger \widehat{a} \ ,$$

and eq. (18). Similarly :

$$\begin{aligned}\left\langle \widehat{X}_2^2 \right\rangle &= -\frac{1}{2} \langle \alpha | \left(\widehat{a}\widehat{a} + \widehat{a}^\dagger \widehat{a}^\dagger - 2\widehat{a}^\dagger \widehat{a} - 1 \right) |\alpha\rangle \\ &= \frac{1}{2} \left\{ -\alpha^2 - (\alpha^*)^2 + 2\alpha^* \alpha + 1 \right\} \\ &= \frac{1}{2} \left(\frac{\alpha - \alpha^*}{i} \right)^2 + \frac{1}{2} = 2\Im \{\alpha\}^2 + \frac{1}{2} \ .\end{aligned} \tag{30b}$$

D. Calculate $\Delta X_i = \sqrt{\langle X_i^2 \rangle_\alpha - \langle X_i \rangle_\alpha^2}$ for $i = 1, 2$.

Solution : Combining the results of eqs.(29) and (30) we find

$$\Delta X_1 = \Delta X_2 = \frac{1}{\sqrt{2}} \ . \tag{31}$$

The take away message here is that quantum "uncertainties" of the quadrature variables of a coherent state are symmetric and the same as for the vacuum state. This is part of the reasoning that has led to coherent states sometimes being referred to as a "displaced" vacuum states.

For the following questions, let us consider a beam splitter with an input state, $|\Psi\rangle$, expressed either in the basis of entry channels $|\Psi\rangle_{1,2}$ or in the basis of output channels $|\Psi\rangle_{3,4}$:

One can interpret the beam splitter as acting on the destruction operators in the following manner :

$$\widehat{a}_3 = r\widehat{a}_1 + t\widehat{a}_2 \qquad \widehat{a}_4 = t\widehat{a}_1 + r'\widehat{a}_2 \ ,$$

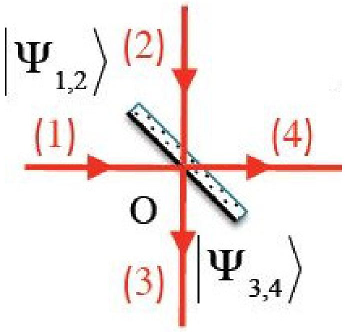

FIGURE 3 – Beam splitter acting translate input channels (1) and (2) into output channels (3) and (4).

where r and t are the **complex valued** reflection and transmission coefficients of the beam splitter (obtained by either measurement or an electromagnetic optics calculation) :

$$\begin{bmatrix}\widehat{a}_3\\ \widehat{a}_4\end{bmatrix} = \begin{bmatrix} r & t \\ t & r'\end{bmatrix}\begin{bmatrix}\widehat{a}_1\\ \widehat{a}_2\end{bmatrix} = [S]\begin{bmatrix}\widehat{a}_1\\ \widehat{a}_2\end{bmatrix} , \tag{32}$$

where $[S]$ is called the S-matrix. In the case of a lossless beam splitter, the S-matrix must be a unitary matrix, *i.e.*

$$S^{\dagger}.S = S.S^{\dagger} = \mathbb{I} . \tag{33}$$

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14. Show that the lossless condition of eq. (33) requires the 'obvious' energy conservation relations :

$$|r|^2 + |t|^2 = 1 \tag{34a}$$
$$|r'|^2 + |t|^2 = 1 \tag{34b}$$

Interpret the significance of these relations in terms of beam power (intensity).

Solution : Writing out explicitly eq.(33) yields :

$$S^{\dagger}.S = \begin{bmatrix} r^* & t^* \\ t^* & r'^{,*}\end{bmatrix}\begin{bmatrix} r & t \\ t & r'\end{bmatrix} = \begin{bmatrix} |r|^2+|t|^2 & r^*t+t^*r' \\ t^*r+t^*r'^{,*} & |r'|^2+|t|^2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} , \tag{35}$$

and so eqs.(34) are obtained by equating the diagonal elements in (35). Since intensity is proportional to the field amplitude squared, (34a) is just the statement that the field intensity in output channel 3 is the sum of input intensity *reflected* from channel 1 plus the intensity *transmitted* from input channel 2. Analogously, (34b) is that the field intensity in output channel 4 is the sum of input intensity *transmitted* from channel 1 plus the intensity *reflected* from input channel 2. We thus conclude that eqs.(34) are the energy conservation relations that one would naievely expect.

15. Show that the lossless condition of eq. (33) also requires the 'less obvious' energy conservation relations :

$$r^*t + t^*r' = 0 \tag{36a}$$
$$t^*r + tr'^{,*} = 0 \tag{36b}$$

Explain why these 'two' conditions are in fact the same condition on the complex coefficients.

Solution : These relations are obtained by equating the off-diagonal elements of eq.(35). The physical interpretation of these constraints is far less obvious than for the previous question. Furthermore, they are in fact only a single condition since eq.(36a) and eq.(36b) are complex conjugates of one another.

16. For a beam splitter that is mirror symmetric with respect to a plane at the center of the beam splitter's diagonal, often called a *symmetric* beam splitter, one has $r = r'$. For this symmetric case, what do the relations of eq. (36) have to say about the phase relations between the coefficients r and t ? (Hint : write $r = |r|e^{i\phi_r}$ and $t = |t|e^{i\phi_t}$, and determine the relation between ϕ_r and ϕ_t imposed by eq. (36).

Solution :

$$\begin{aligned} r^* t = -t^* r' &\implies |r|e^{-i\phi_r}|t|e^{i\phi_t} = -|t|e^{-i\phi_t}|r|e^{i\phi_r} \\ &\implies e^{2i(\phi_t-\phi_r)} = -1 = e^{i\pi} \\ &\implies |\phi_t - \phi_r| = \frac{\pi}{2} \,. \end{aligned} \tag{37}$$

The interesting thing here is that a combination of the mirror symmetry and the unitarity requirement *imposes* that complex transmission and reflection coefficients of the beam splitter be 90° out of phase. This will have important consequences as we will see in the following exercises.

17. Which of these S-matrices are physically acceptable for a lossless beam splitter ?

Solution : We need to verify that the unitarity condition, $S^\dagger = S^{-1}$, of eq.(33) is satisfied for each potential S-matrix.

A. $[S] = \begin{bmatrix} i\rho & t \\ t & i\rho \end{bmatrix}$ for real-valued ρ and t with $\rho^2 + t^2 = 1$.

Solution : Acceptable because $S^\dagger S = \begin{bmatrix} -i\rho & t \\ t & -i\rho \end{bmatrix} \begin{bmatrix} i\rho & t \\ t & i\rho \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We see that the $r = i\rho$ and $t = \tau$ coefficients are indeed out of phase by 90° as required by eq.(37). We were free to take the coefficients t to be real here since the S-matrix is only determined up to a phase factor, but eq.(37) then required here the r coefficient to be purely imaginary.

B. $[S] = \begin{bmatrix} r & t \\ t & -r \end{bmatrix}$ for r and t real-valued with $r^2 + t^2 = 1$.

Solution : Acceptable because $S^\dagger S = \begin{bmatrix} r & t \\ t & -r \end{bmatrix} \begin{bmatrix} r & t \\ t & -r \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. For this S-matrix we have gone back to eq. (32) and *not required* the S-matrix to be mirror symmetric. This allows us to set $r' = -r$ and then fix all the S-matrix coefficients to be real-valued. This S-matrix is often used to model a partially silvered beam splitter with a thin metallic coating on one face. In contrast to typical thin film beam splitters which only work in a narrow band of frequencies, such a beam splitter operates over a much larger spectral range. This S-matrix can be convenient for calculations since r, $-r$ and t are all real numbers, and we can be relatively care-free when calculating adjoints, since $S^\dagger = S$, but we must now be careful with the signs when carrying out calculations. Many books on quantum optics tend to favor calculations with a mirror symmetric beam splitter. As a counter example, the Corserra course by Alain Aspect carries out all beam splitter calculations using the S-matrix presented here. As you might suspect, the measurable physical predictions of such different choices are largely (or exactly) the same.

C. $[S] = \begin{bmatrix} \rho & i\tau \\ -i\tau & \rho \end{bmatrix}$ for ρ and τ real-valued with $\rho^2 + \tau^2 = 1$.

Solution : Not acceptable because $S^\dagger S = \begin{bmatrix} \rho & i\tau \\ -i\tau & \rho \end{bmatrix} \begin{bmatrix} \rho & i\tau \\ -i\tau & \tau \end{bmatrix} = \begin{bmatrix} 1 & 2i\rho\tau \\ -2i\rho\tau & 1 \end{bmatrix}$. The matrix in question here is Hermitian, not unitary.

D. $[S] = \begin{bmatrix} r & i\tau \\ i\tau & r \end{bmatrix}$ for r and τ real-valued with $r^2 + \tau^2 = 1$ for ρ and τ real with $r^2 + \tau^2 = 1$ **Solution :** Acceptable because $S^\dagger S = \begin{bmatrix} r & -i\tau \\ -i\tau & r \end{bmatrix} \begin{bmatrix} r & i\tau \\ i\tau & r \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This is another common way of describing a thin film beam splitter like that described by the S-matrix of A. The choice between A. and D. is just considered a matter of personal choice.

18. Given the S-matrix relation of eq. (32), and the results of previous questions, show that transformation of the creation operators is :

$$\begin{bmatrix} \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{bmatrix} = \begin{bmatrix} r & t \\ t & r' \end{bmatrix} \begin{bmatrix} \hat{a}_3^\dagger \\ \hat{a}_4^\dagger \end{bmatrix} = [S] \begin{bmatrix} \hat{a}_3^\dagger \\ \hat{a}_4^\dagger \end{bmatrix} \,. \tag{38}$$

Solution : We first multiply eq.(32) by $S^\dagger = S^{-1}$:

$$\begin{bmatrix} r^* & t^* \\ t^* & r'^{,*} \end{bmatrix} \cdot \begin{bmatrix} \hat{a}_3 \\ \hat{a}_4 \end{bmatrix} = \begin{bmatrix} r^* & t^* \\ t^* & r'^{,*} \end{bmatrix} \cdot \begin{bmatrix} r & t \\ t & r' \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} \,,$$

$$\implies \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} = \begin{bmatrix} r^* & t^* \\ t^* & r'^{,*} \end{bmatrix} \cdot \begin{bmatrix} \hat{a}_3 \\ \hat{a}_4 \end{bmatrix}$$

$$\implies \begin{cases} \hat{a}_1 = r^* \hat{a}_3 + t^* \hat{a}_4 \\ \hat{a}_2 = t^* \hat{a}_3 + r'^{,*} \hat{a}_4 \end{cases} \,,$$

Taking the adjoint of all quantities in the this last equation leads to :

$$\begin{aligned}\widehat{a}_1^\dagger &= r\widehat{a}_3^\dagger + t\widehat{a}_4^\dagger \\ \widehat{a}_2^\dagger &= t\widehat{a}_3^\dagger + r'\widehat{a}_4^\dagger\end{aligned} \implies \begin{bmatrix}\widehat{a}_1^\dagger \\ \widehat{a}_2^\dagger\end{bmatrix} = \begin{bmatrix} r & t \\ t & r'\end{bmatrix}\begin{bmatrix}\widehat{a}_3^\dagger \\ \widehat{a}_4^\dagger\end{bmatrix} \qquad \text{Q.E.D.} \tag{39}$$

19. Let us consider the state $|n\rangle_1$ with n photons in channel 1. Express $|n\rangle_1$ in terms of $\widehat{a}_1^\dagger$ and the vacuum state of channel 1, $|0\rangle_1$:
   - **A.** $|n\rangle_1 = \frac{1}{\sqrt{n!}}\left(\widehat{a}_1^\dagger\right)^n|0\rangle_1$
   - B. $|n\rangle_1 = \frac{1}{\sqrt{(n-1)!}}\left(\widehat{a}_1^\dagger\right)^n|0\rangle_1$
   - C. $|n\rangle_1 = \sqrt{n!}\left(\widehat{a}_1^\dagger\right)^n|0\rangle_1$

   Solution : Result A. is correct since $|n\rangle$ is produced by applying, $\widehat{a}_1^\dagger$, n times to the vacuum state and appealing to the fact that $\widehat{a}_1^\dagger|n\rangle_1 = \sqrt{n+1}\,|n+1\rangle_1$ on the state $|0\rangle_1$. The answers B. and C. are not correct because only answer A. satisfies the normalization condition that ${}_1\langle n|n\rangle_1 = 1$.

20. Express $|\Psi\rangle = |n\rangle_1 \otimes |m\rangle_2 = |n, m\rangle_{1,2}$ in terms of $\widehat{a}_1^\dagger$ and $\widehat{a}_2^\dagger$ acting on the vacuum of the 2 channels $|0, 0\rangle$. Note that the vacuum state doesn't depend on the chosen basis. (more than one correct response possible)
   - **A.** $|\Psi\rangle = \frac{1}{\sqrt{n!m!}}\widehat{a}_1^{\dagger n}\widehat{a}_2^{\dagger m}|0,0\rangle$
   - B. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}}\widehat{a}_1^{\dagger m}\widehat{a}_2^{\dagger n}|0,0\rangle$
   - **C.** $|\Psi\rangle = \frac{1}{\sqrt{n!m!}}\widehat{a}_2^{\dagger m}\widehat{a}_1^{\dagger n}|0,0\rangle$

   Solution : The tensor product space of n photons states in input channel 1 and m photons in channel 2 is produced by multiplying $\widehat{a}_1^{\dagger n}$ by $\widehat{a}_2^{\dagger m}$ with these operators acting on the vacuum state $|0, 0\rangle$ (**n.b.** we wrote the vacuum state, $|0,0\rangle = |0\rangle_1|0\rangle_2 = |0\rangle_3|0\rangle_4$ in order to insist that the vacuum state is a tensor product of vacuum states in channels or 1 and 2, or alternatively 3 and 4, which for simplicity one often writes simply $|0\rangle$). The factor $\frac{1}{\sqrt{n!m!}}$ assures the proper normalisation. Both A. and C. are correct here, since that $\widehat{a}_1^{\dagger n}$ and $\widehat{a}_2^{\dagger n}$ commute, their order is irrelevant.

21. Consider the state $|\Psi\rangle = |1, 0\rangle_{1,2}$. Give the expression for this state in the base of the output channels.
   - A. $|\Psi\rangle = |1,0\rangle_{1,2} = r^*|1,0\rangle_{3,4} + t^*|0,1\rangle_{3,4}$
   - B. $|\Psi\rangle = |1,0\rangle_{1,2} = t^*|1,0\rangle_{3,4} + r^*|0,1\rangle_{3,4}$
   - C. $|\Psi\rangle = |1,0\rangle_{1,2} = r|1,0\rangle_{3,4} + t|0,1\rangle_{3,4}$

   Solution : Using eq.(39) to express $\widehat{a}_1^\dagger$ in terms of $\widehat{a}_3^\dagger$ and $\widehat{a}_4^\dagger$, we obtain the result :

   $$|1,0\rangle_{1,2} = \widehat{a}_1^\dagger|0,0\rangle = \left(r\widehat{a}_3^\dagger + t\widehat{a}_4^\dagger\right)|0,0\rangle = r|1,0\rangle_{3,4} + t|0,1\rangle_{3,4}\ .$$

22. Find the probability of detecting a photon in channels (3) and (4) respectively when a 1-photon state is incident in channel (1)? Concretely this amounts to calculate the probability with Born's rule ($P_3 = |{}_{3,4}\langle 1,0|\Psi\rangle|^2$ and $P_4 = |{}_{3,4}\langle 0,1|\Psi\rangle|^2$). Explain why results like these lead to people sometimes saying that Maxwell equations can be interpreted being a 'quantum' theory of 1-photon states.

   Solution : The probability of detecting the photon in 'perfect' detector (3) is the same as calculating the probability for finding $|\Psi\rangle$ in the state $|1,0\rangle_{3,4}$:

   $$\begin{aligned}P_3 = |{}_{3,4}\langle 1,0|\Psi\rangle|^2 &= |{}_{3,4}\langle 1,0|\left\{r|1,0\rangle_{3,4} + t|0,1\rangle_{3,4}\right\}|^2 \\ &= |r|^2\ ,\end{aligned}$$

   and in the same manner,

   $$\begin{aligned}P_4 = |{}_{3,4}\langle 0,1|\Psi\rangle|^2 &= |{}_{3,4}\langle 0,1|\left\{r|1,0\rangle_{3,4} + t|0,1\rangle_{3,4}\right\}|^2 \\ &= |t|^2\ .\end{aligned}$$

   We recall that the 'classical' prediction of Maxwells equations for the intensity of field in outputs (3) and (4) are respectively :

   $$\begin{aligned}I_3 &= I|r|^2 \\ I_4 &= I|t|^2\end{aligned}$$

Recalling that $I \propto ||\boldsymbol{E}||^2$, we see that the field 'intensity' predictions of Maxwell equations can be equated to the probability of a 1-photon state being detected. This is all quite reassuring since quantum optics has to contain the same physics as classical electromagnetism, and if look at a classical field as corresponding to a high flux of individual photons, then the field intensity at a point in an optical system will be proportional to the quantum probability of finding a photon at the space-time point in question.

The quantum theory of single photons does however make predictions that a classical theory *cannot* describe, perhaps the most notable being the fact that it predicts that a single-photon sent into a beam splitter must be detected in either detector (3) or detector (4) but not both at the same time. At this point, it may look like quantum optics only makes rather '*minor*' corrections to classical electromagnetism. The next questions show however that when one considers a second photon, dramatic departures from classical physics become possible.

23. Express the state $|\Psi\rangle = |n,m\rangle_{1,2}$ in terms of the creation operators in the output channels, $\hat{a}_3^\dagger$ and $\hat{a}_4^\dagger$ for a mirror symmetric beam splitter where $r = r'$ in the S-matrix of eq.(32).
   A. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \left(r^*\hat{a}_3^\dagger - t^*\hat{a}_4^\dagger\right)^n \left(t^*\hat{a}_3^\dagger + r^*\hat{a}_4^\dagger\right)^m |0,0\rangle$
   B. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \left(r^*\hat{a}_3^\dagger + t^*\hat{a}_4^\dagger\right)^n \left(t^*\hat{a}_3^\dagger + r^*\hat{a}_4^\dagger\right)^m |0,0\rangle$
   C. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \left(r\hat{a}_3^\dagger + t\hat{a}_4^\dagger\right)^n \left(t\hat{a}_3^\dagger + r\hat{a}_4^\dagger\right)^m |0,0\rangle$

   Solution : This result is obtained by employing the relations of eq. (38) in the expression given in question 20.

24. Consider the state, $|\Psi\rangle = |1,1\rangle_{1,2}$, corresponding to exactly 1-photon arriving simultaneously in each entree mode channel on a mirror symmetric beam splitter. What is the correct expression for $|\Psi\rangle = |1,1\rangle_{1,2}$ in the output basis ?
   A. $|\Psi\rangle = \sqrt{2}\, rt|2,0\rangle_{3,4} + \left[t^2 + r^2\right]|1,1\rangle_{3,4} + \sqrt{2}\, tr|0,2\rangle_{3,4}$

   Solution : This result follows from 23.C because one has :

$$\begin{aligned} |\Psi\rangle = |1,1\rangle_{1,2} &= \left(r\hat{a}_3^\dagger + t\hat{a}_4^\dagger\right)\left(t\hat{a}_3^\dagger + r\hat{a}_4^\dagger\right)|0,0\rangle \\ &= rt\hat{a}_3^\dagger\hat{a}_3^\dagger|0,0\rangle + r^2\hat{a}_3^\dagger\hat{a}_4^\dagger|0,0\rangle + t^2\hat{a}_4^\dagger\hat{a}_3^\dagger|0,0\rangle + tr\hat{a}_4^\dagger\hat{a}_4^\dagger|0,0\rangle \\ &= \sqrt{2}\, rt|2,0\rangle_{3,4} + \left[t^2 + r^2\right]|1,1\rangle_{3,4} + \sqrt{2}\, tr|0,2\rangle_{3,4} \end{aligned}$$

   B. $|\Psi\rangle = \sqrt{2}\, r^*t^*|2,0\rangle_{3,4} + \left[(t^*)^2 + (r^*)^2\right]|1,1\rangle_{3,4} + \sqrt{2}\, t^*r^*|0,2\rangle_{3,4}$
   C. $|\Psi\rangle = r^*t^*|2,0\rangle_{3,4} + \left[(t^*)^2 + (r^*)^2\right]|1,1\rangle_{3,4} + t^*r^*|0,2\rangle_{3,4}$

25. Let us continue with the preceding question with the state $|\Psi\rangle = |1,1\rangle_{1,2}$ corresponding to exactly one photon in each entry channel. Consider now a symmetric 50/50 beam splitter with $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. What is the probability to detect precisely 1 photon in each output mode channel ? (This is the famous Hong-Ou-Mandel effect)
   A. 0
   B. 1/3
   C. 1/4
   D. 1

   Solution : $P(1,1) = \left|{}_{4,3}\langle 1,1|\Psi\rangle\right|^2 = \left|{}_{4,3}\langle 1,1|1,1\rangle_{1,2}\right|^2 = \left|{}_{4,3}\langle 1,1|\left[t^2+r^2\right]|1,1\rangle_{3,4}\right|^2 = 0$, where the 90° phase difference between r and t is crucial here because it is what gives us that $t^2 + r^2 = 0$. It is worth remaking that the probability of observing a 2-photon state in output state (3), $P(2,0)$, is :

$$P(2,0) = \left|{}_{4,3}\langle 2,0|\Psi\rangle\right|^2 = \left|{}_{4,3}\langle 2,0|1,1\rangle_{1,2}\right|^2 = \left|{}_{4,3}\langle 2,0|\sqrt{2}rt|2,0\rangle_{3,4}\right|^2 = \frac{1}{2} \tag{40a}$$

   and likewise the probability of observing a 2-photon state in output state (4), $P(0,2)$, is :

$$P(0,2) = \left|{}_{4,3}\langle 0,2|\Psi\rangle\right|^2 = \left|{}_{4,3}\langle 0,2|1,1\rangle_{1,2}\right|^2 = \left|{}_{4,3}\langle 0,2|\sqrt{2}tr|2,0\rangle_{3,4}\right|^2 = \frac{1}{2}\,. \tag{40b}$$

   We thus see that probabilities are indeed conserved and one only observes photons that have 'coalesced' into a 2-photon state in one of the output channels. This experiment led to a Nobel Prize due to its convincing demonstration of a the truly quantum behavior of light.

26. Consider a partially silvered beam splitter described by the S-matrix in 17.B . Demonstrate that this beam splitter generates the same coalesced 2-photon states as obtained with the symmetric beam splitter.